Ex 1.2.8 (b) A function $f: \mathbb{N} \to \mathbb{N}$ that is onto, not 1-1.

(find ex or show impossible)

Let. $f(x) = |x-5|+1$

$f(\{1,2,3,4,5,6,7,8,\dots\}) = (5,4,3,2,1,2,\dots)$

f is not 1-1, because $f(1) = |1-5|+1 = 5$

and $f(9) = |9-5|+1 = 5$ ✓

Scratch

$\forall y \in \mathbb{N}$, $|x-5|+1 = y$ ← have to be able to solve for x

$|x-5| = y-1$

$x-5 = \pm(y-1)$ ← pick + ✓

$x = y-1+5 = y+4$

For any $y \in \mathbb{N}$, let $x = y+4 \in \mathbb{N}$,

and $f(x) = |(y+4)-5|+1$. Since $y+4-5 = y-1 \geq 0$,

$f(x) = (y+4)-5+1 = y$.

$\therefore$ f is onto. ✓

$\therefore$ f is an example of an onto, not 1-1 fcn from $\mathbb{N} \to \mathbb{N}$.

(Give example or show impossible) (c) A function $f: \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Let $g(1) = 0$
$g(2) = 1$
$g(3) = -1$
$g(4) = 2$
$g(5) = -2$
$\vdots$

$g(n) = \ldots \Leftarrow$ figure out formula.

g is a 1-1 corresp (1-1 and onto).

Ex. 1.2.10 Prove or disprove

ⓐ Let $a, b \in \mathbb{R}$. Then $a < b \iff a < b + \varepsilon \ \forall \varepsilon > 0$.

(i) ($\Rightarrow$) If $a < b$, then $\forall \varepsilon > 0$, $a < b + \varepsilon$.

(ii) ($\Leftarrow$) If $\forall \varepsilon > 0$, $a < b + \varepsilon$, then $a < b$.

(ii) Is False. Let $a = 1.79 = b$.

Then $\forall \varepsilon > 0 \quad a = 1.79 < b + \varepsilon = 1.79 + \varepsilon$

(by adding $0 < \varepsilon$ to $a = b$.)

But it is not true that $a < b$. $\blacksquare$.

(This provides a counterexample.)

(i) Suppose $a < b$. Then $\forall \varepsilon > 0$,
Then $0 < \varepsilon$, so $a + 0 < b + \varepsilon$. $\blacksquare$
$\Rightarrow a < b + \varepsilon$.

(c) Let $a, b \in \mathbb{R}$. Then $a \leq b \Leftrightarrow$

$\forall \varepsilon > 0,\ a < b + \varepsilon$.

(i) If $a \leq b$, then $\forall \varepsilon > 0$, $a < b + \varepsilon$. $(\Rightarrow)$

(ii) $(\Leftarrow)$ If $\forall \varepsilon > 0$, $a < b + \varepsilon$, then $a \leq b$.

(i) If $a \leq b$ and $0 < \varepsilon$, then adding inequalities,

True

$a < b + \varepsilon$. ✓

(another way to see: $a \leq b = b + 0 < b + \varepsilon$

so $a < b + \varepsilon$.)

(ii) (By contradiction)

Suppose $a > b$. Then let

$\varepsilon = \frac{a-b}{2}$, and

$$b + \varepsilon = b + \frac{a-b}{2}$$

$$= \frac{a}{2} + \frac{b}{2} = \frac{a+b}{2} < \frac{a+a}{2}$$

$$= a.$$

So $b + \varepsilon < a \Rightarrow b + \varepsilon \leq a$

so $\exists \varepsilon > 0$ s.t. $a \geq b + \varepsilon$.

$\therefore$ By contrapositive, the original statement

$\forall \varepsilon > 0, a < b + \varepsilon \Longrightarrow a \leq b$. $\square$

Note:

$\neg (A \rightarrow B)$

$\equiv A \wedge \neg B$

$A \rightarrow B$

$\equiv \neg B \rightarrow \neg A$

Sets of Real #s, Axioms of Completeness.

Let $S \subseteq \mathbb{R}$ (let S be a subset of $\mathbb{R}$)

We say S is bounded above if $\exists M \in \mathbb{R}$ such that $x \leq M$ $\forall x \in S$.

(We would say M is an upper bound for S.)

We say S is bounded below if $\exists L \in \mathbb{R}$ s.t. $y \geq L$ $\forall y \in S$.

(We say L is a lower bound for S.)

Example $A = (-1, 3] = \{x \in \mathbb{R} : -1 < x \leq 3\}$.

5 is an upper bound for A, since $\forall x \in A$, $x \leq 3 < 5$.

-1 is ~~an~~ lower bound for A

Because $\forall y \in A$, $-1 < y$.

Let $B = \mathbb{N} \cup (-1, 3]$.

-1 is a lower bound of B, but B has no upper bound.

If an upper bound P of $S \subseteq \mathbb{R}$ is contained in S, then we say that P is the maximum of S.

... minimum ...

Note: $A = (-1, 3]$ has a maximum (3) but no minimum.

We say that x is a least upper bound of a set $S \subseteq \mathbb{R}$ if

ⓐ x is an upper bound for S

ⓑ If y is any upper bound for S, then $x \leq y$.

If the least upper bound exists, $x = \text{lub } S = \sup S$

$\uparrow$ supremum

We say that x is a greatest lower bound of a set $S \subseteq \mathbb{R}$ if

ⓐ x is an lower bound for S

ⓑ If y is any lower bound for S, then $y \leq x$.

If the greatest lower bound exists, $x = \text{glb } S = \inf S$

$\uparrow$ infimum

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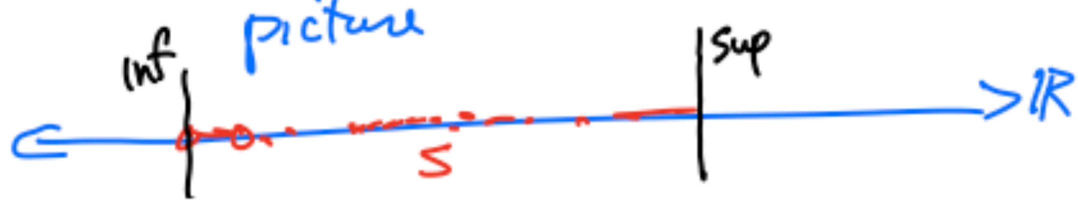
picture



If $\sup S$ exists and $\sup S \in S$,
we call it the maximum of S.

If $\inf S$ exists and $\inf S \in S$,
we call it the minimum of S.

Axiom of Completeness (AOC). (bdd above $\Rightarrow$ sup exists)

Every nonempty set S in $\mathbb{R}$ that is bounded from above has a least upper bound.

(Note: would be equivalent to
bdd below $\Rightarrow$ inf exists)

Example. 1.3.8 Find the sup & inf, if they exist.

(a) $\left\{ \frac{m}{n} : m, n \in \mathbb{N}, m < n \right\} = A$

Ans. $0 = \inf A$, $1 = \sup A$.

Proof: 0 is a lower bound, because if
$m, n \in \mathbb{N}$, $m < n$
$\frac{m}{n} > 0$. ✓

Suppose y is any lower bound for A.

Then $y < \frac{1}{n}$ $\forall n \geq 2$ (letting $m=1$).

(By Archimedean principle, if $y>0$, can find n s.t. $y > \frac{1}{n}$ $\therefore$ $y \leq 0$.)

$\therefore$ 0 is inf.

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1 is a upper bd.

$m, n \in \mathbb{N}$

$m < n$

$\Rightarrow \frac{m}{n} < \frac{n}{n} = 1$

$\Rightarrow$ 1 is an upperbound.

Also $\frac{m}{m+1} \in A$ $\forall m \in \mathbb{N}$

$$\frac{m}{m+1} = \frac{m+1}{m+1} - \frac{1}{m+1} = 1 - \frac{1}{m+1}.$$

We can show that if $y < 1$, then $\exists m \in \mathbb{N}$ s.t. $1 - \frac{1}{m+1} > y$, so y can't be an upper bound.

So any upper bound y must satisfy $y \geq 1$.

$\therefore$ 1 is the sup A.

Lemma Given a set $S \subseteq \mathbb{R}$ that is bounded from above, the supremum of S is unique.

(So we can call it <u>the</u> sup S.).

<u>Proof:</u> If b_1 & b_2 are both sups of S, then they are both upperbounds.
Since b_1 is a sup, $b_1 \le b_2$.
Since b_2 is a sup, $b_2 \le b_1$
$\therefore b_1 = b_2$ $\therefore$ The sup is unique. $\boxtimes$